9.2, 9.3 WS/Quiz solutions

1 Problem 1

1. Find the following limits:

$$\text{(a) } \lim_{n\to\infty} \frac{5/n-3n^2}{2/n-4n^2} \quad \text{(b) } \lim_{k\to\infty}(2k)^{3/k} \quad \text{(c) } \lim_{n\to\infty} n\tan(\pi/n)$$

(a) $\lim_{n\to\infty} \frac{5/n-3n^2}{2/n-4n^2} = \lim_{n\to\infty} \frac{-3n^2}{-4n^2} = 3/4$ since $\lim_{n\to\infty} 5/n = 0 = \lim_{n\to\infty} 2/n$. We also could have applied L'Hopital's rule twice to the form ∞/∞.

(b) $\lim_{k\to\infty}(2k)^{3/k}$: Note that $\lim_{k\to\infty} 2^{1/k} = 1 = \lim_{k\to\infty} k^{1/k}$ from our examples, so

$$\lim_{k\to\infty}(2k)^{1/k} = \lim_{k\to\infty} 2^{1/k}\cdot k^{1/k} = 1 = \lim_{k\to\infty} 2^{1/k}\cdot \lim_{k\to\infty} k^{1/k}$$

by properties of finite limits. Then, since $(2k)^{3/k} = (2^{1/k}\cdot k^{1/k})^3$, again by properties of finite limits, we have

$$\lim_{k\to\infty}(2k)^{3/k} = (\lim_{k\to\infty} 2^{1/k}\cdot k^{1/k})^3 = 1^3 = 1$$

(c) $\lim_{n\to\infty} n\tan(\pi/n)$: here we have an indeterminate form $\infty\cdot 0$. We can use a trick here with the fact that $\lim_{x\to 0}\sin(x)/x = 1$:

$$\begin{aligned}
\lim_{n\to\infty} n\tan(\pi/n) = \lim_{n\to\infty}\frac{\tan(\pi/n)}{1/n} &= \lim_{n\to\infty}\frac{\sin(\pi/n)}{1/n\cdot\cos(\pi/n)} &(1)\\
&= \lim_{n\to\infty}\frac{\pi\cdot\sin(\pi/n)}{\pi/n\cdot\cos(\pi/n)} &(2)\\
&= \lim_{m\to 0}\sin(m)/m\cdot\lim_{n\to\infty}\pi/\cos(\pi/n) &(3)\\
&= 1\cdot\pi/1 = \pi &(4)
\end{aligned}$$

Where we multiplied by $\pi/\pi = 1$ in the limit to go from (1)$\to$ (2). We could have tried using L'Hopital's rule for $0/0$ on $\frac{\tan(\pi/n)}{1/n}$ but it would get hairy.

2 Problem 2

2. Let $a_0 = 1, a_1 = 1+1/3$, $a_2 = 1+1/3+1/9$, $a_3 = 1+1/3+1/9+1/27$, and so on. For an arbitrary positive integer n, write down the formula for a_n. Then find $\lim_{n\to\infty} a_n$ (hint: use the formula $1-r^{n+1} = (1+r+r^2+...+r^n)(1-r)$ for real r and for $n = 1,2,3,..$).

Note that we have $a_2 = 1+(1/3)+(1/3)^2, a_3 = 1+(1/3)+(1/3)^2+(1/3)^3$, so $a_n = 1+(1/3)+...+(1/3)^n$. Then, by the formula

$$1-(1/3)^{n+1} = (1+(1/3)+...+(1/3)^n)(1-(1/3)) = a_n\cdot 2/3$$

we have $a_n = 3/2(1-(1/3)^{n+1})$. Since $\lim_{n\to\infty}(1/3)^n = 0$, we have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3/2(1-(1/3)^{n+1}) = 3/2$$

3 Problem 3

You deposit \$100 in a savings account that pays 5% interest compounded annually. Thus after 1 year there is the original \$100 plus the interest $100(0.05)$ dollars in the account, that is, there are $100+100(0.05)$ dollars in the account.

(a) Show that after 2 years there are $100(1+0.05)^2$ dollars in the account.

Let Y_1 be the amount in the account after one year, so $Y_1 = 100+100(0.05)$. The amount in the account after 2 years, Y_2, is the amount from Y_1 plus 5% of that amount, i.e. 5% of $Y_1 = 100+100(0.05)$, which is $0.05(100+100(0.05))$. Then we have

$$\begin{aligned} Y_2 = Y_1 + 0.05 \cdot Y_1 &= 100 + 100(0.05) + 0.05(100 + 100(0.05)) && (5)\\ &= 100 + 100(0.05) + 100(0.05) + 100(0.05)^2 && (6)\\ &= 100 + 2 \cdot 100(0.05) + 100(0.05)^2 && (7)\\ &= 100(1 + 2(0.05) + 0.05^2) && (8)\\ &= 100(1+0.05)^2 && (9) \end{aligned}$$

and we're done.

(b) Let n be an arbitrary positive integer. Find a formula for the amount in the account after n years.

Well, in general, the amount Y_n in the account after n years is the amount in the account from the previous year (i.e. after $n-1$ years) plus 5% interest of that amount. Thus

$$Y_n = Y_{n-1} + 0.05Y_{n-1} = Y_{n-1}(1.05).$$

Since

$$Y_{n-1} = Y_{n-2}(1.05),$$

we have

$$Y_n = Y_{n-1}(1.05) = Y_{n-2}(1.05)^2,$$

and in general

$$Y_n = Y_{n-i}(1.05)^i$$

for any $i \leq n$. For $i = n$, then, we have $Y_n = Y_0(1.05)^n$, where $Y_0 = 100$ is the initial amount in the account.

Thus a formula for the amount in the account after n years is

$$Y_n = 100(1.05)^n.$$

(c) Determine how many years it would take for the amount in the account to reach \$200.

We need to find the first year that $Y_n \geq 200$, i.e. find n so that $100(1.05)^n \geq 200$, so $(1.05)^n \geq 2$.

Since ln is an increasing function, we have $\ln(1.05^n) \geq \ln 2$, so $n \ln(1.05) \geq \ln 2$. Then

$$n \geq \ln 2 / \ln(1.05) \approx 14.2.$$

Since n is an integer, we'll have $n = 15$ years is the first year the account will reach \$200.

4 Problem 4

When a superball is dropped onto a hardwood floor, it bounces up to approximately 80% of its original height. Suppose that the ball is dropped initially from a height of $5m$ above the floor, and let $b_n =$ the maximum height of the n'th bounce.

(a) Evaluate b_1, b_2, b_3.

The max height of the first bounce, b_1, will be 80% of 5m, i.e. $b_1 = 0.8 \cdot 5 = 4$ meters.

The max height of the second bounce, b_2, will be 80% of the max height of the first bounce. Thus $b_2 = 0.8 \cdot b_1 = 0.8 \cdot 4 = 3.2$ meters

Similarly, $b_3 = 0.8 \cdot b_2 = 0.8 \cdot 3.2 = 2.4$ meters.

(b) Prove that $\lim_{n\to\infty} b_n = 0$

Note that

$$b_n = 0.8 \cdot b_{n-1} = 0.8 \cdot (0.8 \cdot b_{n-2}) = 0.8^2 \cdot b_{n-2} = ... = 0.8^n b_0 = 0.8^n \cdot 5$$

so $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 0.8^n \cdot 5$. Note $\lim_{n\to\infty} 0.8^n = 0$ since $0.8 < 1$, and $\lim_{n\to\infty} 5 = 5$. Thus $\lim_{n\to\infty} 0.8^n \cdot 5 = \lim_{n\to\infty} 0.8^n \cdot \lim_{n\to\infty} 5 = 0 \cdot 5 = 0$ by properties of finite limits. Thus $\lim_{n\to\infty} b_n = 0$.

5 Problem 5

(a) Suppose that $\lim_{n\to\infty} a_n = L$. Tell in a complete sentence why $\lim_{n\to\infty} a_{n+1} = L$

(b) Suppose that $\lim_{n\to\infty} a_n = 3$. Is it necessarily true that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$? Explain why it is true, or why it is not necessarily true (hint: do we know $\lim_{n\to\infty} \frac{a_n}{b_n}$?)

(c) Suppose $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \pi$. Prove that $\lim_{n\to\infty} a_n b_n = 0$.

(a) Suppose that $\lim_{n\to\infty} a_n = L$. Tell in a complete sentence why $\lim_{n\to\infty} a_{n+1} = L$

Note that the sequence $\{a_{n+1}\}_{n\geq 1}$ is $a_2, a_3, ...$, which is $\{a_n\}_{n\geq 1}$ without the first term: so

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$$

since the limit does not depend on the first term of the sequence a_1.

(b) Suppose that $\lim_{n\to\infty} a_n = 3$. Is it necessarily true that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$? Explain why it is true, or why it is not necessarily true (hint: do we know $\lim_{n\to\infty} \frac{a_n}{b_n}$?)

It is true since if $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n \neq 0$ both are finite, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$. Since $\lim_{n\to\infty} a_n = 3 \neq 0$, this property of finite limits applies:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\lim_{n\to\infty} a_{n+1}}{\lim_{n\to\infty} a_n} = 3/3 = 1.$$

(c) Suppose $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \pi$. Prove that $\lim_{n\to\infty} a_n b_n = 0$.

Again, by properties of finite limits, if $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = \pi$, then

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = 0 \cdot \pi = 0.$$

6 Problem 6

(a) Show that

$$\begin{aligned} \ln(n+1) - \ln n &= \int_1^{n+1} \frac{1}{t}\, dt - \int_1^n \frac{1}{t}\, dt && (10)\\ &= \int_n^{n+1} \frac{1}{t}\, dt \geq \frac{1}{n+1} && (11) \end{aligned}$$

Note that for any integral,

$$(\max_{[a,b]} f(x))(b-a) \geq f(x) \int_a^b f(x)\, dx \geq (\min_{[a,b]} f(x)) \cdot (b-a)$$

since an integral represents the area under the graph of f: the area of the rectangle determined by the minimum value of f on $[a,b]$ and the length $(b-a)$ of $[a,b]$ is smaller than this total area.

Thus

$$\int_n^{n+1} \frac{1}{t}\, dt \geq (\min_{[n,n+1]} 1/t) \cdot (n+1-n) = 1/(n+1)$$

since $1/t$ is a decreasing function on $(0,\infty)$.

(b) Let $a_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln n$ for $n = 1, 2, ...$ Use (a) to show $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence (hint: show $a_n - a_{n+1} \geq 0$).

We show $a_n - a_{n+1} \geq 0$. Note

$$a_n - a_{n+1} = [1+1/2+...+1/n - \ln n] - [1+1/2+...+1/n+1/(n+1) - \ln(n+1)] = \ln(n+1) - \ln(n) - 1/(n+1).$$

We just showed $\ln(n+1) - \ln(n) \geq 1/(n+1)$, so we have $\ln(n+1) - \ln(n) - 1/(n+1) \geq 0$. Thus $a_n - a_{n+1} \geq 0$, so $a_n \geq a_{n+1}$ and $\{a_n\}_{n\geq 1}$ is a decreasing sequence.

(c) Using the left sum of $\int_1^{n+1} \frac{1}{t}\, dt$ with partition $\{1, 2, ..., n+1\}$, show that $1 + \frac{1}{2} + ... + \frac{1}{n} \geq \ln(n+1)$.

The left sum of $\int_a^b f\, dx$ on the partition $(a = a_0, ..., a_n = b)$ is $f(a_0)(a_1 - a_0) + ... + f(a_{n-1})(a_n - a_{n-1})$, so the left sum of $\int_1^{n+1} \frac{1}{t}\, dt$ is

$$(1/1)(2-1) + (1/2)(3-2) + ... + (1/n)(n+1-n) = 1 + 1/2 + ... + 1/n$$

Also note that the left sum of a decreasing function is greater than the integral of the function on the interval (a_0, a_n). Thus, since $1/t$ is decreasing on $(0, \infty)$, we have

$$1 + 1/2 + ... + (1/n) \geq \int_1^{n+1} 1/t\, dt = [\ln|t|]_1^{n+1} = \ln(n+1) - \ln 1 = \ln(n+1)$$

Thus $1 + 1/2 + ... + 1/n \geq \ln(n+1)$ and we're done.

(d) Use (c) and the definition of a_n to show that $a_n \geq 0$ for all n.

Note $a_n = 1 + 1/2 + ... + 1/n - \ln n$ and since $1 + 1/2 + ... + 1/n \geq \ln(n+1)$, we have

$$1 + 1/2 + ... + 1/n - \ln(n+1) \geq 0$$

.

Note also that the natural logarithm is an increasing function, $\ln(n+1) > \ln n$ for all integers $n > 0$. Thus $a_n = 1 + 1/2 + ... + 1/n - \ln n > 1 + 1/2 + ... + 1/n - \ln(n+1) \geq 0$, so a_n is positive for all n.

(e) Use (b) and (d) to show that $\{a_n\}_{n=1}^{\infty}$ converges to a number r ($r \approx 0.577216$ and is known as the Euler Mascheroni constant).

Since $\{a_n\}$ is a decreasing sequence with $a_n > 0$ for all n, it converges by Theorem 9.6 in our book (a bounded sequence that is either increasing or decreasing converges).

7 Quiz 9.2

Evaluate the limit as a number, ∞ or $-\infty$:

$$\text{(a) } \lim_{k\to\infty} \sqrt[k]{2k} \quad \text{(b) } \lim_{n\to\infty} \ln(\tfrac{1}{n}) \quad \text{(c) } \lim_{n\to\infty} \tfrac{n+3}{n^2-2}$$

(a) $\lim_{k\to\infty} \sqrt[k]{2k}$

Note $\lim_{k\to\infty} \sqrt[k]{2} = 1 = \lim_{k\to\infty} \sqrt[k]{k}$ from our examples, so since $\sqrt[k]{2k} = \sqrt[k]{2} \cdot \sqrt[k]{k}$, we have

$$\lim_{k\to\infty} \sqrt[k]{2k} = \lim_{k\to\infty} \sqrt[k]{2}\sqrt[k]{k} = \lim_{k\to\infty} \sqrt[k]{k} \cdot \lim_{k\to\infty} \sqrt[k]{2} = 1$$

by properties of finite limits.

(b) $\lim_{n\to\infty} \ln(\frac{1}{n})$

This is the same as $\lim_{x\to 0} \ln(x)$ since $\lim_{n\to\infty} 1/n = 0$, and since the natural logarithm is continuous. Since $\lim_{x\to 0} \ln x = -\infty$, we have $\lim_{n\to\infty} \ln(1/n) = -\infty$

(c) $\lim_{n\to\infty} \frac{n+3}{n^2-2}$

We can ignore the constant terms since as $n \to \infty$ they make a negligible contribution (in general, with polynomials, we can ignore lower degree terms in both the numerator and denominator):

$$\lim_{n\to\infty} \frac{n+3}{n^2-2} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$$

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(Bonus) Find the first 3 digits of $\int_{-2}^{2} (x^3 \cos(\frac{x}{2}) - 1/2)\sqrt{4-x^2}\, dx$

Expanding the integrand, we have

$$\int_{-2}^{2} (x^3 \cos(\frac{x}{2}) - 1/2)\sqrt{4-x^2}\, dx = \int_{-2}^{2} x^3 \cos(\frac{x}{2})\sqrt{4-x^2} - 1/2\sqrt{4-x^2}\, dx$$

We'll use the properties of function evenness and oddness: f is even if $f(-x) = f(x)$ for all x in the domain, and f is odd if $f(-x) = -f(x)$ for all x in the domain. If f is even, $\int_{-a}^{a} f(x) = 2 \cdot \int_0^a f(x)\, dx$ and if f is odd, $\int_{-a}^{a} f(x)\, dx = 0$.

Note that $x^3 \cos(\frac{x}{2})\sqrt{4-x^2}$ is an odd function since it is the product of the even function $\cos(\frac{x}{2})\sqrt{4-x^2}$ (since both $\cos(x/2)$ and $\sqrt{4-x^2}$ are even) with the odd function x^3. Thus it integrates to zero over the symmetric interval $[-2, 2]$

We also know that $\frac{1}{2}\sqrt{4-x^2}$ is an even function, (i.e. so

$$\int_{-2}^{2} -\frac{1}{2}\sqrt{4-x^2}\, dx = 2 \cdot \int_0^2 -\frac{1}{2}\sqrt{4-x^2}\, dx = -\int_0^2 \sqrt{4-x^2}\, dx,$$

which is the area of a quarter of the circle of radius 2 (the graph of $\sqrt{4-x^2}$ is the semicircle of radius 2 in the first and second quadrants, centered at the origin). The full area is $\pi 2^2 = 4\pi$, so a quarter of this area is π.

Thus

$$\int_{-2}^{2} (x^3 \cos(\frac{x}{2}) - 1/2)\sqrt{4-x^2}\, dx = -\pi$$

which has first three digits -3.14.